LÉVY PROCESSES - LECTURE 6

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Some proofs are omitted in the notes since their presentation in the book [1] is very clear but still lengthy. I hope to be able to present them during the lecture. The lecture consist of material from p. 62-65 in Chapter 1, p. 83 in Chapter 2 and 143-152, 163-165 in Chapter 3 of [1]. Material from [3] and [2] has been used too.

1. Preparation

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. A family $(\mathcal{F}_t)_{t\in T}$ of sub sigma fields of $\mathcal{F}$ is called a *filtration* if for all $s, t \in T$ with $s < t$

$$\mathcal{F}_s \subset \mathcal{F}_t.$$

Let $(X(t))_{t\in T}$ be a stochastic process on a $(\Omega, \mathcal{F}, \mathbf{P})$. Then X is called *adapted* to the filtration $(\mathcal{F}_t)_{t\in T}$ if $X(t)$ is $\mathcal{F}_t$ measurable for every $t \in T$.

2. Some Markov theory for Lévy processes

Let $\mathcal{B}_b(\mathbf{R}^d)$ denote the space of all bounded Borel functions equipped with the uniform norm. A stochastic process $(X(t))_{t\in T}$ is called a Markov process if for all $f \in \mathcal{B}_b(\mathbf{R}^d)$ and all $s < t$

$$\mathbf{E}[f(X(t))|\mathcal{F}_s] = \mathbf{E}[f(X(t))|X(s)].$$

In words this means that if we want to predict the future, given all information up to time s, then we can forget what happened before that time.

Lemma 2.1. *Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. If X and Y are $\mathbf{R}^d$-valued random variables such that X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$ then*

$$\mathbf{E}[f(X,Y)|\mathcal{G}] = G_f(X), \quad \text{a.s.}$$

for all $f \in \mathcal{B}_b(\mathbf{R}^{2d})$, where $G_f(x) = \mathbf{E}[f(x,Y)]$ for each $x \in \mathbf{R}^d$.

Proposition 2.2. *Every Lévy process is a Markov process.*

Proof. Because of the independent increments and the Lemma

$$\mathbf{E}[f(X(t))|\mathcal{F}_s] = \mathbf{E}[f((X(t) - X(s)) + X(s))|\mathcal{F}_s] = G_f(X(s))$$

where

$$G_f(x) = \mathbf{E}[f(X(t) - X(s) + x)].$$

We have that

$$G_f(X(s)) = \mathbf{E}[G_f(X(s))|X(s)] = \mathbf{E}[f(X(t) - X(s) + X(s))|X(s)] = \mathbf{E}[f(X(t))|X(s)]$$

and we are done. □

To every Lévy process we relate two families $(p_t)_{t\in T}$ and $(T_t)_{t\in T}$ of measures and operators respectively. For a fixed $t \in T$ we let $p(t)$ be the measure corresponding to the law of $X(t)$, i.e., $p(t) = X(t)^{-1} \circ \mathbf{P}$. The operator T_t on $\mathcal{B}_b(\mathbf{R}^d)$ is defined by the action

$$(T_t f)(x) = \mathbf{E}[f(X(t) + x)] = \int_{\mathbf{R}^d} f(y + x) p_t(\,\mathrm{d}y) = \int_{\mathbf{R}^d} f(y) p_t(\,\mathrm{d}y - x).$$

We will prove that both the families $(p_t)_{t\in T}$ and $(T_t)_{t\in T}$ are semigroups with respect to convolution and composition respectively. Given one of $(X(t))_{t\in T}$, $(p_t)_{t\in T}$ or $(T_t)_{t\in T}$ one can construct the others.

From $(T_t)_{t\in T}$ we construct $(p_t)_{t\in T}$ as follows:

$$p_t(A) = (T_t\chi_A)(0) \quad \Big(= \mathbf{P}(X(t)\in A|X(0)=0)\Big).$$

There is an issue of measurability here. The Markov process X and its semigroup $(T_t)_{t\in T}$ are called *normal* if T_t maps $\mathcal{B}_b(\mathbf{R}^d)$ into itself, for every $t \in T$. This holds if and only if the map $x \mapsto p_t(A-x)$ is measurable for every $t\in T$ and $A \in \mathcal{B}_{\mathbf{R}^d}$. We say that $(p_t)_{t\in T}$ is *normal* if so is the case. One class of normal homogenous Markov processes is the Feller processes. We will prove this together with the fact that every Lévy processes is Feller. First we will explore the relationship between the Lévy process $(X(t))_{t\in T}$ and its convolution semigroup $(p_t)_{t\in T}$.

2.1. **The convolution semigroup.** A family $(p_t)_{t\in T}$ of probability measures is said to converge weakly to δ_0 if

$$\lim_{t\to 0}\int_{\mathbf{R}^d} f(y)\,\mathrm{d}p_t(y) = f(0)$$

for all $f \in C_b(\mathbf{R}^d)$.

Proposition 2.3. *If X is a stochastic process wherein $X(t)$ has law p_t for each $t \geq 0$ and $X(0) = 0$ (a.s.) then $(p_t)_{t\in T}$ is weakly continuous to δ_0 if and only if X is stochastically continuous at $t = 0$.*

Proof. See Proposition 1.4.1 in [1]. □

A family $(p_t)_{t\in T}$ of probability measures is said to be a convolution semigroup if for every $s, t \in T$

$$p_{s+t} = p_s * p_t,$$

meaning that for every $A \in \mathcal{B}_b(\mathbf{R}^d)$,

$$p_{s+t}(A) = \int_{\mathbf{R}^d} p_s(A-x)p_t(\,\mathrm{d}x) = \int_{\mathbf{R}^d} p_t(A-x)p_s(\,\mathrm{d}x) = \int_{\mathbf{R}^{2d}} \chi_A(x+y)p_s(\,\mathrm{d}x)p_t(\,\mathrm{d}y).$$

Proposition 2.4. *If X is a Lévy process wherein $X(t)$ has law p_t for each $t\in T$ then $(p_t)_{t\in T}$ is a weakly continuous convolution semigroup.*

Proof. Since X is a Lévy process it satisfies by (L1) $X(0) = 0$ a.s. and by (L3) the stochastic continuity. By proposition 2.3 the family $(p(t))_{t\in T}$ is therefore weakly continuous. Since by (L2) $X(s+t) = X(s+t) - X(s) + X(s)$ and $X(s+t) - X(s), X(s)$ are independent with the laws p_t, p_s respectively we have that $p_{s+t} = p_s * p_t$. □

Theorem 2.5. *If $(p(t))_{t\in T}$ is a weakly continuous convolution semigroup of probability measures, then there exists a Lévy process X such that, for each $t \in T$, $X(t)$ has law $p(t)$.*

Proof [1],[2]. Let $\Omega = \{\omega\colon T \to \mathbf{R}^d, \omega(0) = 0\}$. Let $\mathcal{C}$ denote the algebra of cylinder sets of the form

$$I^{A_1,\dots,A_n}_{t_1,\dots,t_n} = \{\omega\colon \omega(t_1)\in A_1,\dots,\omega(t_n)\in A_n\},$$

where $t_1,\dots,t_n \in T$, $A_1,\dots,A_n \in \mathcal{B}_{\mathbf{R}^d}$ and $n \in \mathbf{N}$. Let $\mathcal{F}$ be the σ-algebra generated by $\mathcal{C}$. We will define a set function $\mathbf{P}$ on $\mathcal{C}$ in order to get a consistent family of finite dimensional distributions for the process. Let

$$\begin{aligned}\mathbf{P}(I^{A_1,\dots,A_n}_{t_1,\dots,t_n}) = &\int_{\mathbf{R}^d}\cdots\int_{\mathbf{R}^d} \chi_{A_1}(y_1)\chi_{A_2}(y_1+y_2)\cdots\chi_{A_n}(y_1+\dots+y_n)\\ &\times p_{t_1}(\,\mathrm{d}y_1)p_{t_2-t_1}(\,\mathrm{d}y_2)\cdots p_{t_n-t_{n-1}}(\,\mathrm{d}y_n)\end{aligned} \tag{2.1}$$

By the Kolmogorov Extension Theorem the set function $\mathbf{P}$ is extended to a measure on $\mathcal{F}$. Define the canonical Lévy process $X(t)(\omega) = \omega(t)$. Then X has the finite dimensional distributions

$$\mathbf{P}(X(t_1)\in A_1,\dots,X(t_n)\in A_n) = \mathbf{P}(I^{A_1,\dots,A_n}_{t_1,\dots,t_n}).$$

and it has univariate laws corresponding to $(p_t)_{t\in T}$. Since, by assumption, this is a convolution semigroup of probability measures converging weakly to δ_0 and by Proposition 2.3, (L1) X(0)=0 and (L3) stochastic continuity are satisfied. The increments are clearly stationary since their laws are determined by the semigroup. It remains to prove that the increments are independent.

We claim that for $f \in \mathcal{B}_b(\mathbf{R}^d)$

$$\mathbf{E}[f(X(t_1), \dots, X(t_n))] = \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} f(y_1, y_1 + y_2, \dots, y_1 + \cdots + y_n) \\ \times p_{t_1}(\,\mathrm{d}y_1) p_{t_2 - t_1}(\,\mathrm{d}y_2) \cdots p_{t_n - t_{n-1}}(\,\mathrm{d}y_n).$$

Notice that for f of the form $f = \chi_{A_1} \cdots \chi_{A_n}$, for $A_1, \dots, A_n \in \mathcal{B}_{\mathbf{R}^d}$, this is nothing but (2.1). By approximation by simple functions and taking the limit we obtain the claimed equality.

Choose

$$f(x_1, \dots, x_n) = \exp\Big(i \sum_{j=1}^n \langle u_j, x_j - x_{j-1} \rangle_{\mathbf{R}^d}\Big).$$

Then

$$\begin{aligned}
&\mathbf{E}\Big[\exp\Big(i \sum_{j=1}^n \langle u_j, X(t_j) - X(t_{j-1}) \rangle\Big)\Big] \\
&= \int_{\mathbf{R}^d} \cdots \int_{\mathbf{R}^d} \exp\Big(i \sum_{j=1}^n \langle u_j, y_j \rangle\Big) p_{t_1}(\,\mathrm{d}y_1) p_{t_2 - t_1}(\,\mathrm{d}y_2) \cdots p_{t_n - t_{n-1}}(\,\mathrm{d}y_n) \\
&= \prod_{j=1}^n \int_{\mathbf{R}^d} \exp(i \langle u_j, y_j \rangle) p_{t_j - t_{j-1}}(\,\mathrm{d}y_j) \\
&= \prod_{j=1}^n \mathbf{E}[\exp(i \langle u_j, X(t_j) - X(t_{j-1}) \rangle)]
\end{aligned}$$

implying independence of increments [1]. □

2.2. **The transition semigroup.** Recall that

$$(T_t f)(x) = \mathbf{E}[f(X(t) + x)] = \mathbf{E}[f(X(t)) | X(0) = x], \quad t \in T, \quad f \in \mathcal{B}_b(\mathbf{R}^d).$$

If $T_t(\mathcal{B}_b(\mathbf{R}^d)) \subset \mathcal{B}_b(\mathbf{R}^d)$ then $(T_t)_{t \in T}$ and $(X(t))_{t \in T}$ are called *normal*.

Theorem 2.6. *If X is a time homogenous normal Markov process, then*

1. *T_t is a linear operators on $\mathcal{B}_b(\mathbf{R}^d)$.*
2. *$T_0 = I$.*
3. *$T_s T_t = T_{s+t}$, for every $s, t \in T$.*
4. *$f \geq 0 \Rightarrow T_t f \geq 0$, for all $t \in T$.*
5. *T_t is a contraction, i.e., $\|T_t\| \leq 1$, for each $t \in T$.*
6. *$T_t(1) = 1$ for all $t \in T$.*

Proof. (1), (2), (4) and (6) are obvious. To prove (3) we use the Markov property of X together with the law of iterated expectation. For $f \in \mathcal{B}_b(\mathbf{R}^d)$

$$\begin{aligned}
(T_{s+t} f)(x) &= \mathbf{E}[f(X(t+s)) | X(0) = x] \\
&= \mathbf{E}[\mathbf{E}[f(X(t+s)) | \mathcal{F}_s] | X(0) = x] \\
&= \mathbf{E}[\mathbf{E}[f(X(t+s)) | X(s)] | X(0) = x] \\
&= \mathbf{E}[(T_t f)(X(s)) | X(0) = x] \\
&= (T_s(T_t f))(x).
\end{aligned}$$

Notice here that we used that T_t is normal since if $x \mapsto (T_t f)(x)$ were not bounded and Borel measurable then the last line would not be well defined.

[1]The elements of a random vectors are independent if and only if its characteristic function is the product of the characteristic functions of its elements

For (5)

$$\begin{aligned}\|T_t f\| &= \sup_{x\in\mathbf{R}^d} |\mathbf{E}[f(X(t))|X(s)=x]| \\ &\le \sup_{x\in\mathbf{R}^d} |\mathbf{E}[|f(X(t))|\big|X(s)=x]| \\ &\le \sup_{x\in\mathbf{R}^d} |f(x)| \\ &= \|f\|\end{aligned}$$

□

A homogenous Markov process is said to be a *Feller process* if

$$T_t(C_0(\mathbf{R}^d)) \subset C_0(\mathbf{R}^d)$$

for all $t \in T$ and

$$\lim_{t\to 0} \|T_t f - f\| = 0$$

for all $f \in C_0(\mathbf{R}^d)$. The last statement is the same as saying that T_t is a strongly continuous semigroup on $C_0(\mathbf{R}^d)$. We say that T_t is a *Feller semigroup* if the same conditions holds, regardless of its connection to X.

Theorem 2.7. *If X is a Feller process, then its transition probabilities p_t are normal.*

Proof [2]. Since T_t is Feller, for every $x \in \mathbf{R}^d$ the map $f \mapsto (T_t f)(x)$ is a positive linear form on $C_0(\mathbf{R}^d)$. By Riesz representation theorem there exists a unique Radon measure $q_t(\cdot, x)$ such that

$$(T_t f)(x) = \int_{\mathbf{R}^d} f(y) q_t(\,\mathrm{d}y, x)$$

for every $f \in C_0(\mathbf{R}^d)$. The map $x \mapsto \int f(y)q_t(\,\mathrm{d}y, x)$ is in $C_0(\mathbf{R}^d)$ by the Feller property and is thus measurable. Take an increasing sequence of functions $(f_n)_{n\in\mathbf{N}} \subset C_0(\mathbf{R}^d)$ such that $f_n \to \chi_A$. Then by a version of the Monotone Class Theorem the mappings

$$x \mapsto \int_{\mathbf{R}^d} f_n(y) q_t(\,\mathrm{d}y, x)$$

converges to a measurable function $x \mapsto q_t(A, x)$. Let $(p_t)_{t\in T}$ be the transition probabilities of X. Since

$$(T_t f)(x) = \int_{\mathbf{R}^d} f(y+x) p_t(\,\mathrm{d}y) \int_{\mathbf{R}^d} f(y) p_t(\,\mathrm{d}y - x)$$

we can identify that $p_t = q_t(\cdot, 0)$ and $p_t(\cdot - x) = q_t(\cdot, x)$. Since q_t is measurable in the right sense, so is p_t. □

Theorem 2.8. *Every Lévy process is a Feller process.*

Proof. See Theorem 3.1.9 in [1]. □

3. Representation of semigroups and generators by pseudo-differential operators

This is Section 3.3.3 in [1]. The presentation is very clear an need no clarifications, by means of lecture notes.

References

[1] D. Applebaum *Lévy Processes and Stochastic Calculus*, Cambridge **2009**.
[2] D. Revuz and M. Yor *Continuous Martingales and Brownian Motion*, Springer **1991**
[3] K.I. Sato *Lévy Processes and Infinitely Divisible Distributions*, Cambridge **1999**.